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Traveling wave solutions to the $(n+1)$-dimensional sinh–cosh–Gordon equation

Xinghua Fan *, Shouxiang Yang, Jiuli Yin, Lixin Tian

Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, China

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ABSTRACT

Traveling wave solutions for a generalized sinh–cosh–Gordon equation are studied. The equation is transformed into an auxiliary partial differential equation without any hyperbolic functions. By using the theory of planar dynamical system, the existence of different kinds of traveling wave solutions of the auxiliary equation is obtained, including smooth solitary wave, periodic wave, kink and antikink wave solutions. Some explicit expressions of the blow-up solution, kink-like solution, antikink-like solution and periodic wave solution to the generalized sinh–cosh–Gordon equation are given. Planar portraits of the solutions are shown.

1. Introduction

In this work, the following $(n+1)$-dimensional generalized combined sinh–cosh–Gordon equation:

$$u_{tt} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \alpha \sinh(mu) + \beta \cosh(mu) = 0 \tag{1}$$

is studied, where α, β are real constants and m is a positive integer. Eq. (1) can be seen as a generalization of the well-studied sinh–Gordon equation

$$u_{tt} - u_{xx} + \sinh u = 0 \tag{2}$$

which is derived from research on the surfaces of positive constant curvature and various areas of physics [1]. There are some other generalized form of (2). By using the bifurcation method, the generalized double sinh–Gordon equation

$$u_{tt} - ku_{xx} + 2\alpha \sinh(mu) + \beta \sinh(mu) = 0, \tag{3}$$

and the generalized sinh–Gordon equation

$$u_{tt} - au_{xx} + b \sinh(nu) = 0 \tag{4}$$

were studied in [2,3], respectively. By using the variable separated ODE and the tanh method, the sinh–cosh–Gordon equation

$$u_{tt} - ku_{xx} + \alpha \sinh u + \beta \cosh u = 0 \tag{5}$$

was studied in [4]. Explicit traveling wave solutions for (3)–(5) were given. There is a growing interest in the study of the high-dimensional sinh–Gordon equation and other equations [5–9] where several kinds of traveling wave solutions were obtained.

* Corresponding author.
E-mail address: fan131@ujs.edu.cn (X. Fan).

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Clearly, when $m = 1$, Eq. (1) becomes (5). We are interested in what role the parameter m plays in (1). Will the type of the traveling wave solution be influenced by it? To answer this question, we consider the dynamical bifurcation behavior for the traveling wave solutions of (1). We shall apply the method of dynamical system [10–13] to find the exact traveling wave solutions of (1).

The rest of this paper is organized as follows. In Section 2, after transforming (1) into a planar system, we discuss the bifurcation conditions and the phase portraits of the planar system, where the parametric conditions are derived. Explicit expressions for the traveling wave solutions are given in Section 3. The last section devotes to a brief summary and some numerical integration results.

2. Bifurcation conditions and possible phase portraits

In this section, the properties of equilibrium points and possible phase portraits will be given.

We look for the traveling wave solutions of (1) in the form of $u(t, x_1, x_2, \ldots, x_n) = u(\sum_{i=1}^{n} k_i x_i - ct) = u(\xi)$, where c is the wave speed and $\xi = \sum_{i=1}^{n} k_i x_i - ct$. Using the Painlevé transformation [4]

$$v = \mathrm{e}^{u}, \tag{6}$$

we have

$$\sinh(mu) = \frac{v^m - v^{-m}}{2}, \qquad \cosh(mu) = \frac{v^m + v^{-m}}{2}. \tag{7}$$

Then (1) becomes

$$2vv_{tt} - 2v_t^2 - 2\sum_{i=1}^{n}(vv_{x_i x_i} - v_{x_i}^2) + \alpha(v^m - v^{-m})v^2 + \beta(v^m + v^{-m})v^2 = 0. \tag{8}$$

Letting $v(t, x_1, x_2, \ldots, x_n) = \phi(\sum_{i=1}^{n} k_i x_i - ct) = \phi(\xi)$, (8) is changed to a nonlinear ODE

$$2\left(c^2 - \sum_{i=1}^{n} k_i^2\right)\phi\phi'' - 2\left(c^2 - \sum_{i=1}^{n} k_i^2\right)(\phi')^2 + (\alpha+\beta)\phi^{2+m} - (\alpha-\beta)\phi^{2-m} = 0, \tag{9}$$

where "$\prime$" is the derivative with respect to ξ.

Let $k = \sum_{i=1}^{n} k_i^2$ and suppose $c^2 \neq k$. Letting $y = \phi'$, we get the following planar system

$$\begin{cases} \dfrac{\mathrm{d}\phi}{\mathrm{d}\xi} = y, \\ 2(k - c^2)\phi\dfrac{\mathrm{d}y}{\mathrm{d}\xi} = (\alpha+\beta)\phi^{2+m} - (\alpha-\beta)\phi^{2-m} + 2(k - c^2)y^2. \end{cases} \tag{10}$$

System (10) is a planar dynamical system defined in a 5-parameter space (α, β, c, k, m). Because the phase orbits defined by the vector fields of system (10) determine all traveling wave solutions, we will investigate the bifurcations of phase portraits of these systems in the phase plane (ϕ, y) as the parameters are changed. Here, we should point out that, if a solution $\phi(\xi)$ of system (10) approaches zero, then $\ln(\phi(\xi))$ approaches $-\infty$. In other words, this solution determines an unbounded traveling wave solution of (1).

System (10) is a singular system with a singular line $\phi = 0$. To avoid the singularity, letting $\mathrm{d}\xi = 2(k - c^2)\phi\,\mathrm{d}\tau$, system (10) is changed to a regular system

$$\begin{cases} \dfrac{\mathrm{d}\phi}{\mathrm{d}\tau} = 2(k - c^2)\phi y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau} = (\alpha+\beta)\phi^{2+m} - (\alpha-\beta)\phi^{2-m} + 2(k - c^2)y^2. \end{cases} \tag{11}$$

System (11) has the same phase portraits as system (10) except for $\phi = 0$.

Suppose that $v = \phi(\xi)$ is a traveling wave solution for (8) for $\xi \in (-\infty, \infty)$, $\lim_{\xi\to-\infty}\phi(\xi) = a$ and $\lim_{\xi\to\infty}\phi(\xi) = b$, where a and b are two constants. If $a = b$, then $\phi(\xi)$ is called a solitary wave solution. If $a \neq b$, then $\phi(\xi)$ is called a kink (or an antikink) solution. Usually, a solitary wave solution for (8) corresponds to a homoclinic orbit of system (10) while a kink (or an antikink) wave solution corresponds to a heteroclinic orbit (or the so-called connecting orbit). Similarly, a periodic solution for (8) corresponds to a closed orbit of system (10).

Now we consider the equilibrium points of system (11). Let $f(\phi) = (\alpha+\beta)\phi^{2+m} - (\alpha-\beta)\phi^{2-m}$. Let $A_i(\phi_i, 0)$ be an equilibrium point of system (11) where $f(\phi_i) = 0$. If $\frac{\alpha-\beta}{\alpha+\beta} > 0$, we can derive three equilibrium points $A_0(0, 0)$, $A_{1,2}(\phi_{1,2}, 0) = \left(\pm\sqrt[2m]{\frac{\alpha-\beta}{\alpha+\beta}}, 0\right)$ of system (11). If $\frac{\alpha-\beta}{\alpha+\beta} \leq 0$, system (11) has only one equilibrium $A_0(0, 0)$.

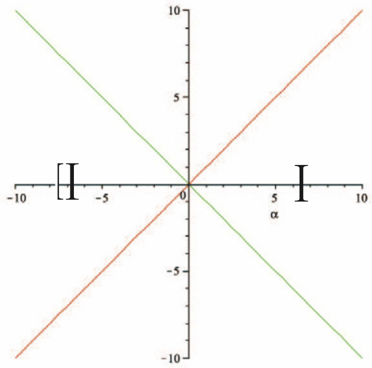


Fig. 1. Bifurcation lines and bifurcation regions.

Let $M(\phi_e, y_e)$ be the coefficient matrix of the linearized system of system (11) at the equilibrium point (ϕ_e, y_e), $J(\phi_e, y_e) = \det M(\phi_e, y_e)$. We have

$$J = 2\left(k - c^2\right)\left(4(k - c^2){y_e}^2 - (2+m)(\alpha+\beta){\phi_e}^{2+m} + (2-m)(\alpha-\beta){\phi_e}^{2-m}\right)$$

$$p(\phi_e, y_e) = \mathrm{Trace}(M(\phi_e, y_e)) = 6\left(k - c^2\right)y_e.$$

For the above three equilibrium points, we have $J(0,0) = p(0,0) = 0$, $J(\phi_1, 0) = (-1)^m 4m(c^2 - k)(\alpha-\beta)\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^{\frac{2-m}{2m}}$, $J(\phi_2, 0) = 4m(c^2 - k)(\alpha-\beta)\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^{\frac{2-m}{2m}}$, $p(\phi_i, 0) = 0$.

By the theory of planar dynamical systems [13], we know that for an equilibrium point of a planar system, if $J < 0$ then it is a saddle; if $J > 0$ and $p = 0$ then it is a center point; if $J = 0$ and the Poincaré index of the equilibrium is zero then it is a cusp.

From the expression of J, we know that equilibrium points A_1 and A_2 have the same type when m is even. When m is odd, one of the two equilibrium points is a center while the other is a saddle point.

We now consider the bifurcation of phase portraits of system (11). Notice that for $\sqrt{\frac{\alpha-\beta}{\alpha+\beta}} = 0$ or $J(\phi_1, 0) = 0$, we have $\alpha = \beta$. For $\phi_2 = \sqrt[2m]{\frac{\alpha-\beta}{\alpha+\beta}}$, we have $\alpha = -\beta$. Therefore, in the (α, β)-parameter plane, we can define the following bifurcation lines:

$$L_1 : \alpha = \beta, \qquad L_2 : \alpha = -\beta. \tag{12}$$

These lines partition the (α, β)-parameter plane into two regions. Bifurcation lines and bifurcation regions in the (α, β)-parameter plane for $c^2 \neq k$ are shown in Fig. 1.

Except for the straight line $\phi = 0$, systems (10) and (11) have the same first integral

$$H(\phi, y) = \frac{(\alpha+\beta)\phi^{2+m} + (\alpha-\beta)\phi^{2-m} + m(c^2 - k)y^2}{m(c^2 - k)\phi^2} = h, \tag{13}$$

where h is an arbitrary constant. From the first integral (13), we get

$$h_{A2} = H(\phi_2, 0) = \frac{2\sqrt{\alpha^2 - \beta^2}}{m(c^2 - k)}, \qquad h_{A1} = H(\phi_1, 0) = (-1)^m h_{A2}. \tag{14}$$

For a fixed h, the level curve $H(\phi, y) = h$ defined by (13) determines a set of invariant curves of system (10), which contains different branches of curves. As h is varied, it defines different families of orbits of system (10) with different dynamical behaviors. Different phase portraits of system (10) for $m = 1, 2, 3, 4$ are shown in Figs. 2 and 3, respectively.

We shall apply these phase portraits to consider the traveling wave solutions of (1) in the next section.

3. Different kinds of traveling solutions of (1)

3.1. Blow-up wave solutions

Proposition 1. *There exists a solitary wave solution with valley form for (8) if $m = 1, c^2 - k > 0, \alpha+\beta < 0, \alpha-\beta < 0$ (or $m = 1, c^2 - k < 0, \alpha+\beta > 0, \alpha-\beta > 0$). Corresponding to this solitary wave solution, $u = \ln(\phi(\xi))$ is a blow-up solution for (1).*

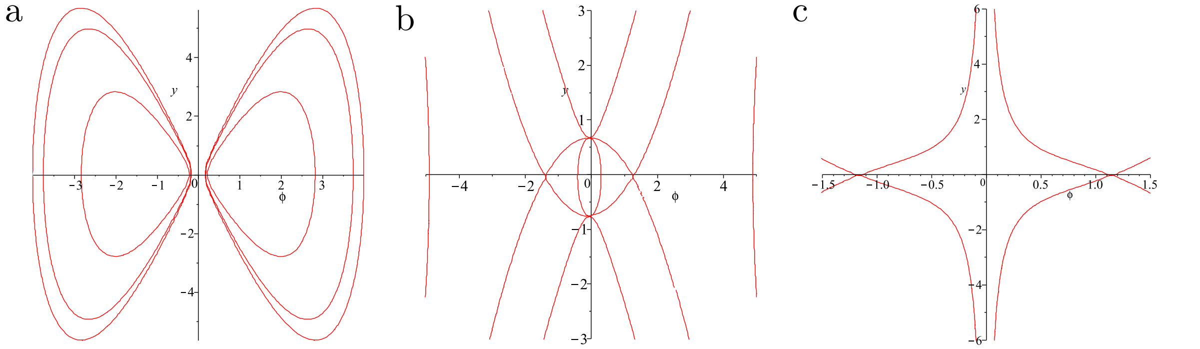


Fig. 2. Phase portraits of (10) for $c^2 \neq k$. (a) $m = 2$ or 4, the case for $(\alpha, \beta) \in$ I, $c^2 - k > 0$ or $(\alpha, \beta) \in$ II, $c^2 - k < 0$. (b) $m = 2$, the case for $(\alpha, \beta) \in$ II, $c^2 - k > 0$ or $(\alpha, \beta) \in$ I, $c^2 - k < 0$. (c) $m = 4$, the case for $(\alpha, \beta) \in$ II, $c^2 - k > 0$ or $(\alpha, \beta) \in$ I, $c^2 - k < 0$.

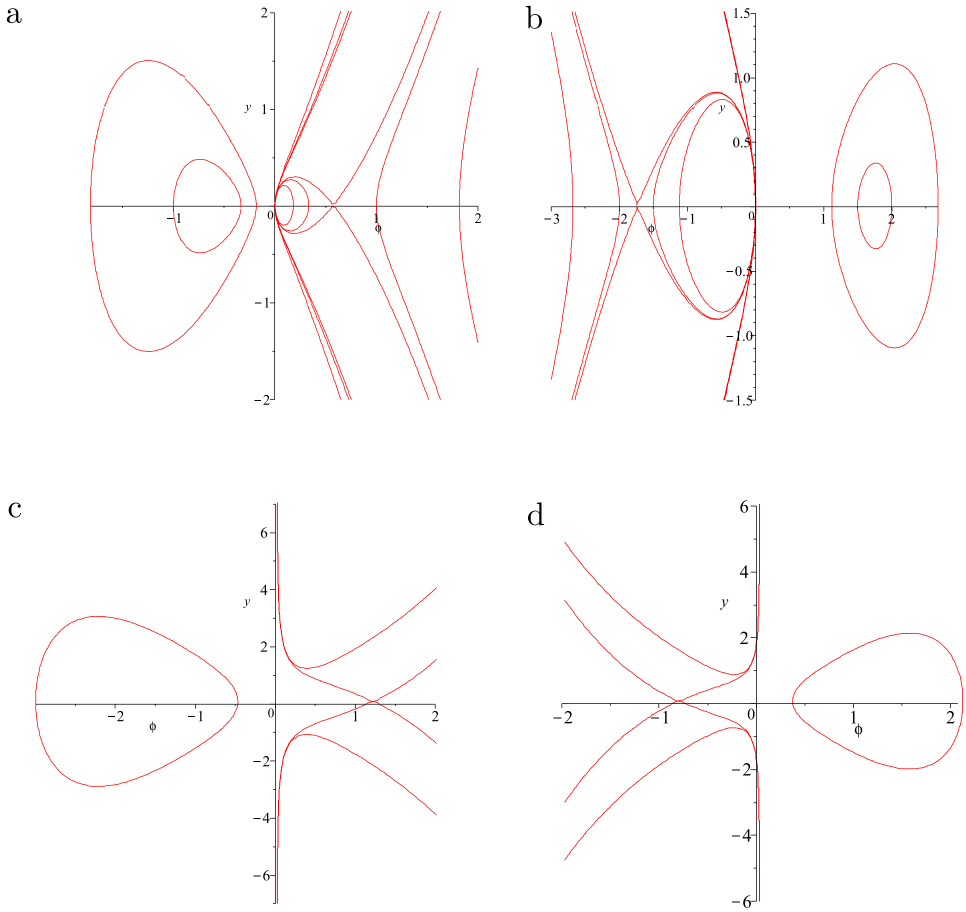


Fig. 3. Phase portraits of (10) for $c^2 \neq k$. (a) $m = 1$, the case for $(\alpha, \beta) \in$ II, $c^2 - k > 0$ or $(\alpha, \beta) \in$ I, $c^2 - k < 0$. (b) $m = 1$, the case for $(\alpha, \beta) \in$ I, $c^2 - k > 0$ or $(\alpha, \beta) \in$ II, $c^2 - k < 0$. (c) $m = 3$, the case for $(\alpha, \beta) \in$ II, $c^2 - k > 0$ or $(\alpha, \beta) \in$ I, $c^2 - k < 0$. (d) $m = 3$, the case for $(\alpha, \beta) \in$ I, $c^2 - k > 0$ or $(\alpha, \beta) \in$ II, $c^2 - k < 0$.

Proof. When $c^2 - k > 0, \alpha + \beta < 0, \alpha - \beta < 0$, as shown in Fig. 3(a), there is a homoclinic orbit to the saddle point $A_2(\phi_2, 0)$ for system (10). The orbit is defined by the following algebraic equation

$$y^2 = \frac{\alpha + \beta}{-c^2 + k}\phi \left(\phi - \sqrt{\frac{\alpha - \beta}{\alpha + \beta}} \right)^2. \tag{15}$$

Substituting (15) into the first equation of system (10) and integrating from $\phi(0) = 0$ to $\phi(\xi)$, we have

$$\int_{\phi(0)}^{\phi(\xi)} \frac{\mathrm{d}\phi}{\sqrt{\phi}\left(\phi - \sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\right)} = \pm\sqrt{\frac{\alpha+\beta}{k-c^2}}\xi. \tag{16}$$

Then we get an exact traveling wave solution of (8) as

$$\phi = \tanh^2\left(\frac{1}{2}\sqrt[4]{\frac{\alpha^2-\beta^2}{(-c^2+k)^2}}\xi\right)\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}, \tag{17}$$

which is a solitary wave solution.

Notice in (17) the solution ϕ goes to zero as ξ tends to zero, then $\ln(\phi(\xi))$ approaches $-\infty$. Thus, we obtain the following representations of a blow-up wave solution of (1).

$$u = \ln\left(\tanh^2\left(\frac{1}{2}\sqrt[4]{\frac{\alpha^2-\beta^2}{(-c^2+k)^2}}\left(\sum_{i=1}^{n} k_i x_i + ct\right)\right)\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\right). \tag{18}$$

Similarly, the case for $c^2 - k < 0, \alpha+\beta > 0, \alpha-\beta > 0$ can be proved. □

Obviously, (17) is a solitary wave solution with valley form. The planar profiles of (17) and (18) are shown in Figs. 4(a) and 5(a), respectively.

Note that the solitary type solution were also obtained by [4]. Our work is to find it in the $(n+1)$-dimensional sense.

3.2. Kink-like solutions and antikink-like solutions

Proposition 2. *There exists a kink-like (antikink-like) solution for* (1) *if* $c^2 - k > 0, \alpha+\beta < 0, \alpha-\beta < 0$ *(or* $c^2 - k < 0, \alpha+\beta > 0, \alpha-\beta > 0$*) whenever* $m = 2, 3$ *or 4.*

Proof. We first see the case $c^2 - k > 0, \alpha+\beta < 0, \alpha-\beta < 0$.

(1) For $m = 2$, there is a heteroclinic orbit connecting the two saddle points $A_1(\phi_1, 0)$ an d$A_2(\phi_2, 0)$, as shown in Fig. 2(b). The orbit is defined by the following algebraic equation

$$y^2 = \frac{\alpha+\beta}{2(k-c^2)}\left(\phi^2 - \sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\right)^2. \tag{19}$$

By using (19) and the first equation of system (10), we have the following expression of traveling wave solution of (8)

$$\phi = \pm\tanh\left(\frac{\sqrt{2}}{2}\sqrt[4]{\frac{\alpha^2-\beta^2}{(-c^2+k)^2}}\xi\right)\sqrt[4]{\frac{\alpha-\beta}{\alpha+\beta}}. \tag{20}$$

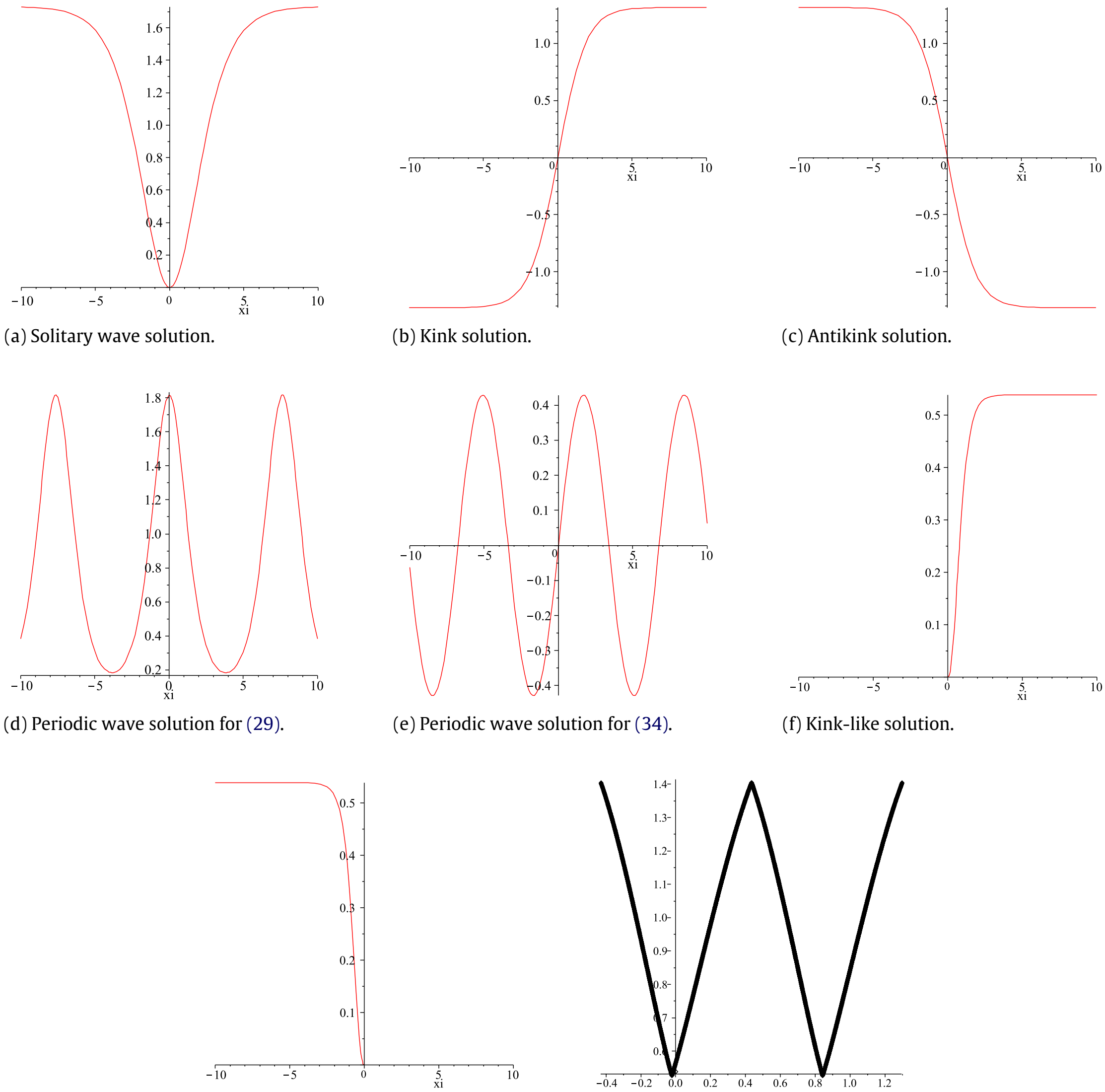
Taking the positive sign (or negative sign), (20) is a kink (or an antikink) wave solution. From (6), we can obtain the following parametric representations of the blow-up wave solution of (1) as

$$u = \ln\left(\pm\tanh\left(\frac{\sqrt{2}}{2}\sqrt[4]{\frac{\alpha^2-\beta^2}{(-c^2+k)^2}}\left(\sum_{i=1}^{n} k_i x_i + ct\right)\right)\sqrt[4]{\frac{\alpha-\beta}{\alpha+\beta}}\right). \tag{21}$$

The planar profiles of (20) and (21) are shown in Fig. 4(b), (c) and Fig. 5(b), (c). Notice that the logarithmic function is defined in the right half line and tanh approaches a constant at the infinity. So we call (21) a kink-like solution (corresponding to the positive sign) or an antikink-like solution (corresponding to the negative sign). Those kinds of solutions were also studied by some authors [14–16].

(2) For $m = 3$, from the phase portrait (Fig. 3(c)), we also see that ϕ is greater than zero. The orbit to the saddle point $A_2(\phi_2, 0)$ of system (10) is not a homoclinic orbit. Compared to the case $m = 1$ (Fig. 3(a)), the homoclinic orbit breaks down to two separate branches. The orbits are defined by the following algebraic equation

$$y^2 = \frac{\alpha+\beta}{3(k-c^2)}\frac{\left(\phi^3 - \sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\right)^2}{\phi}. \tag{22}$$



(a) Solitary wave solution. (b) Kink solution. (c) Antikink solution.

(d) Periodic wave solution for (29). (e) Periodic wave solution for (34). (f) Kink-like solution.

(g) Antikink-like solution. (h) Numerical integration result of periodic solution.

Fig. 4. Different types of planar profiles of solutions to (8).

Substituting (22) into the first equation of system (10) and integrating from $\phi(0) = 0$ to $\phi(\xi)$, we have an exact traveling wave solution of (8) as

$$\phi = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^{\frac{1}{6}} \left(\frac{e^{2L}-1}{e^{2L}+1}\right)^{\frac{2}{3}}, \tag{23}$$

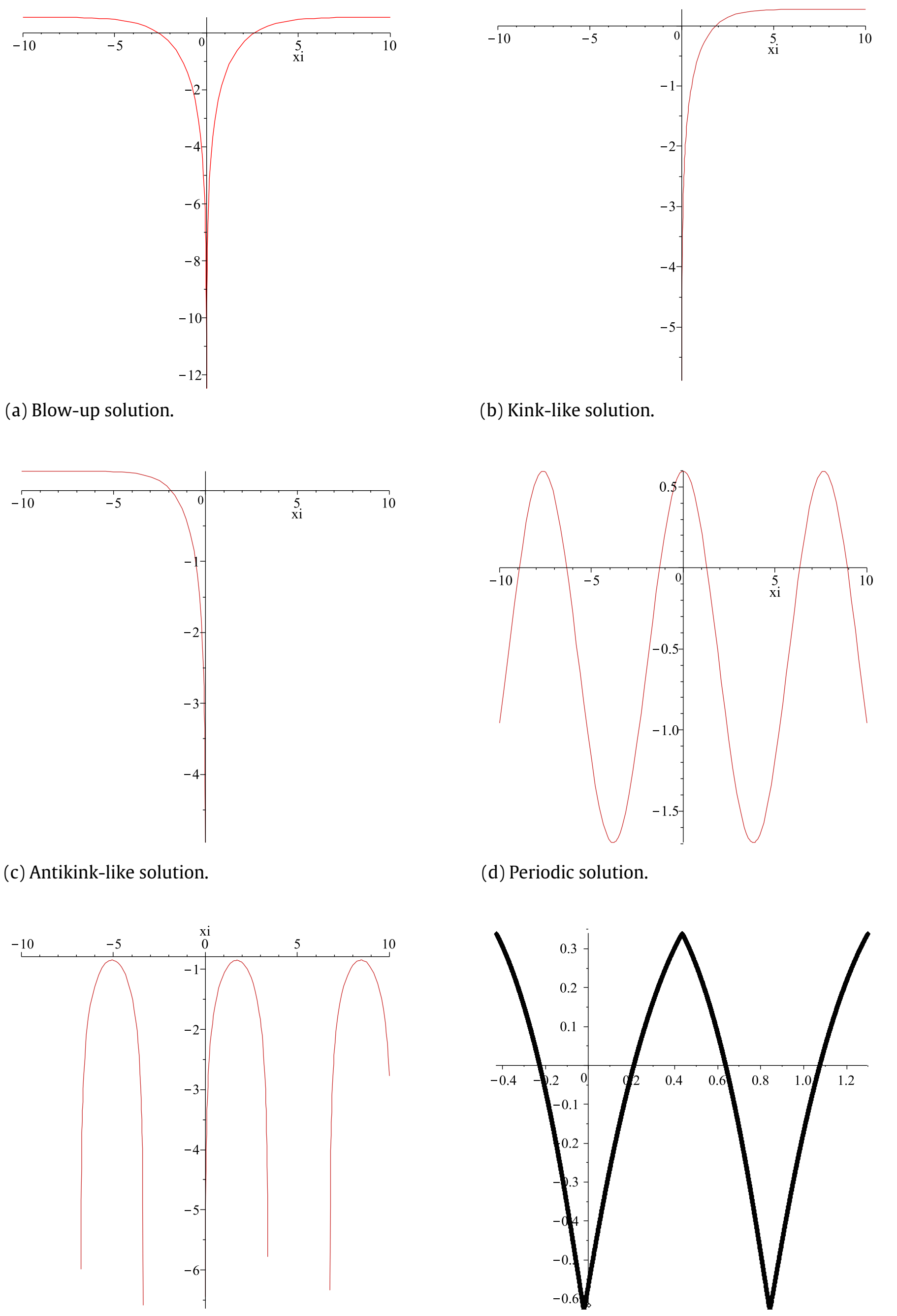
where $L = \pm\frac{\sqrt{3}}{2}\sqrt{\frac{\alpha+\beta}{k-c^2}}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^{\frac{1}{4}}\xi$.

Thus, we obtain the following representations of solution of (1).

$$u = \ln\left(\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^{\frac{1}{6}} \left(\frac{e^{2L}-1}{e^{2L}+1}\right)^{\frac{2}{3}}\right). \tag{24}$$

The planar profiles of (23) and (24) are shown in Figs. 4(f) and 5(b) when L is taken the positive sign, respectively. While Figs. 4(g) and 5(c) correspond to the negative sign.

(3) For $m = 4$, from the phase portrait (Fig. 2(b)), we see that the ϕ is not equal to zero. The orbit connecting the two saddle points $A_1(\phi_1, 0)$ and $A_2(\phi_2, 0)$ is not a heteroclinic orbit. Compared to the case $m = 2$ (Fig. 2(b)), the heteroclinic



(a) Blow-up solution. (b) Kink-like solution.

(c) Antikink-like solution. (d) Periodic solution.

(e) Periodic blow-up solution. (f) Numerical integration result of periodic solution.

Fig. 5. Different types of planar profiles of traveling wave solutions to (1).

orbit breaks down to four separate branches. The orbit is defined by the following algebraic equation

$$y^2 = \frac{\alpha+\beta}{4(k-c^2)} \frac{\left(\phi^4 - \sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\right)^2}{\phi^2}. \tag{25}$$

By using (25) and the first equation of system (10), we have the following expression of traveling wave solution of (8)

$$\phi = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^{\frac{1}{8}} \left(\frac{e^{2L}-1}{e^{2L}+1}\right)^{\frac{1}{2}}, \tag{26}$$

where $L = \pm\sqrt{\frac{\alpha+\beta}{k-c^2}}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^{\frac{1}{4}}\xi$. From (6), we can obtain the following parametric representations of solution of (1).

$$u = \ln\left(\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^{\frac{1}{8}}\left(\frac{e^{2L}-1}{e^{2L}+1}\right)^{\frac{1}{2}}\right). \tag{27}$$

The planar profiles of (26) and (27) are the same as that of (23) and (24), respectively.

Similarly, the cases for $c^2 - k < 0, \alpha+\beta > 0, \alpha-\beta > 0$ can be proved. □

3.3. Periodic wave solutions

Proposition 3. *Corresponding to the closed orbit $H(\phi, y) = h, h \in (h_{A_2}, \infty)$ defined by (13), there exist uncountable many periodic wave solutions for (8) which lead to uncountable many periodic solutions for (1), if $c^2 - k > 0, \alpha+\beta > 0, \alpha-\beta > 0$ (or $c^2 - k < 0, \alpha+\beta < 0, \alpha-\beta < 0$) whenever $m = 1, 3$ or 4.*

Proof. When $c^2 - k > 0, \alpha+\beta > 0, \alpha-\beta > 0, h \in (h_{A_2}, \infty)$, there is a family of periodic orbits enclosing the center $A_2(\phi_2, 0)$ whenever $m = 1, 3$ or 4 (see Fig. 3(b), (d) or Fig. 2(a)).

For $m = 1$ (Fig. 3(b)), each orbit is defined by the following algebraic equation

$$y^2 = \frac{\phi}{k-c^2}((\alpha+\beta)\phi^2 + h(k-c^2)\phi + \alpha - \beta) = \frac{(\alpha+\beta)}{k-c^2}\phi(\phi-A)(\phi-B), \tag{28}$$

where $A = \frac{h(c^2-k)+\sqrt{h^2(c^2-k)^2-4(\alpha^2-\beta^2)}}{2(\alpha+\beta)}$, $B = \frac{h(c^2-k)-\sqrt{h^2(c^2-k)^2-4(\alpha^2-\beta^2)}}{2(\alpha+\beta)}$.

Using (28) and the first equation of system (10), we get an exact traveling wave solution of (8)

$$\phi = A - A\left(sn\left(-\sqrt{\frac{(A-B)(\alpha+\beta)}{4(c^2-k)}}\xi, \sqrt{\frac{A}{A-B}}\right)\right)^2, \tag{29}$$

where $sn(\cdot)$ is the Jacobi elliptic function.

From the phase portrait, we also see that ϕ is greater than zero. Thus, we can obtain the following parametric representations of a smooth periodic traveling wave solution of (1) as

$$u = \ln\left(A - A\left(sn\left(-\sqrt{\frac{(A-B)(\alpha+\beta)}{4(c^2-k)}}\xi, \sqrt{\frac{A}{A-B}}\right)\right)^2\right). \tag{30}$$

The planar profiles of (29) and (30) are shown in Fig. 4(d) and Fig. 5(d), respectively.

Similarly, for $m = 3$ (Fig. 3(d)), the solution is expressed as

$$\int_{\phi(0)}^{\phi(\xi)} \frac{\sqrt{\phi}\,d\phi}{\sqrt{(A-\phi^3)(\phi^3-B)}} = \pm\sqrt{\frac{\alpha+\beta}{3(c^2-k)(\alpha-\beta)}}\xi, \tag{31}$$

where $A = \frac{-3h(k-c^2)+\sqrt{9h^2(k-c^2)^2-4(\alpha-\beta)^2}}{2(\alpha-\beta)}$, $B = \frac{-3h(k-c^2)-\sqrt{9h^2(k-c^2)^2-4(\alpha-\beta)^2}}{2(\alpha-\beta)}$, $A > B$.

For $m = 4$ (Fig. 2(a)), the solution is expressed as

$$\int_{\phi(0)}^{\phi(\xi)} \frac{\phi^2\,d\phi}{\sqrt{(A-\phi^4)(\phi^4-B)}} = \pm\sqrt{\frac{\alpha+\beta}{4(c^2-k)(\alpha-\beta)}}\xi, \tag{32}$$

where $A = \frac{-2h(k-c^2)+\sqrt{4h^2(k-c^2)^2-(\alpha-\beta)^2}}{2(\alpha-\beta)}$, $B = \frac{-2h(k-c^2)-\sqrt{4h^2(k-c^2)^2-(\alpha-\beta)^2}}{2(\alpha-\beta)}$, $A > B$.

For (31) and (32), the numerical integration is taken. Their planar profiles are of the same shape despite scaling. We only show the profile of (31) in Fig. 4(h). Similarly, planar profiles of u corroding to (31) and (32) are of the same shape, too (see Fig. 5(f)).

The case for $c^2 - k < 0, \alpha+\beta < 0, \alpha-\beta < 0$ can be proved in a similar way. □

Note that the periodic wave solutions were not reported in [4].

3.4. Periodic blow-up wave solutions

Proposition 4. *Corresponding to the closed orbit $H(\phi, y) = h, h \in (h_{A_2}, \infty)$ defined by (13), there exist uncountable many periodic wave solutions for (8) which lead to uncountable many periodic blow-up solutions for (1), if $m = 2, c^2-k > 0, \alpha+\beta > 0, \alpha-\beta > 0$ (or $c^2-k < 0, \alpha+\beta < 0, \alpha-\beta < 0$).*

Proof. When $c^2 - k > 0, \alpha + \beta > 0, \alpha - \beta > 0, h \in (h_{A_2}, \infty)$, as shown in Fig. 2(a), there is a family of periodic orbits enclosing the center $A_2(\phi_2, 0)$. Each orbit is defined by the following algebraic equation

$$y^2 = \frac{(\alpha+\beta)(\phi^2 - A)(\phi^2 - B)}{2(k-c^2)}, \tag{33}$$

where $A = \frac{h(c^2-k)+\sqrt{h^2(c^2-k)^2-(\alpha^2-\beta^2)}}{(\alpha+\beta)}, B = \frac{h(c^2-k)-\sqrt{h^2(c^2-k)^2-(\alpha^2-\beta^2)}}{(\alpha+\beta)}, A > B > 0.$

By (33) and the first equation of system (10), we get

$$\phi = \pm sn\left(\frac{\sqrt{2}}{2}\sqrt{\frac{B(\alpha+\beta)}{c^2-k}}\xi, \sqrt{\frac{A}{B}}\right)\sqrt{A}, \tag{34}$$

which is a periodic wave solution.

Note that (34) can approach zero. Thus, we can obtain the following parametric representations of the periodic blow-up wave solution of (1).

$$u = \ln\left(\pm sn\left(\frac{\sqrt{2}}{2}\sqrt{\frac{B(\alpha+\beta)}{c^2-k}}\left(\sum_{i=1}^{n} k_i x_i + ct\right), \sqrt{\frac{A}{B}}\right)\sqrt{A}\right). \tag{35}$$

The case for $c^2 - k < 0, \alpha + \beta < 0, \alpha - \beta < 0$ can be proved similarly. □

The planar profiles of (34) and (35) are shown in Fig. 4(e) and Fig. 5(e).

4. Summary

In this study, we have derived several kinds of traveling wave solutions for (8) including solitary, kink (or antikink) and periodic wave solutions. There are periodic traveling wave solutions whenever $m = 1, 2, 3, 4$. For $m = 1$, there are solitary wave solutions. While for $m = 2, 3, 4$, there are kink (or antikink) wave solution. Solitary wave solution and kink (or antikink) solution do not appear simultaneously. However, cases for other values of m remain to be further researched.

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